Reg. No.....................

Sub. Code: PMAM13

Code No:6833

M.Sc. (CBCS) DEGREE EXAMINATION, APRIL 2021

FIRST SEMESTER

MATHEMATICS - CORE

ANALYTIC NUMBER THEORY

(for those who joined in July 2017 onwards)

Time : Three hours

Maximum: 75 marks

Part - A (10 X 1 = 10 marks)

Answer all question, choose the correct answer:

1. Gcd(-6, -10) = (a) -2 (b) -10 (c) -6 (d) none of the above
2. If $(a, b) = 1$, then $(a^n, b) =$ (a) 1 (b) a (c) b (d) none of the above
3. $\sum_{d/1155} \mu(d) =$ (a) 1 (b) 0 (c) 6 (d) none of the above
4. $(\mu * \phi)(8) =$ (a) 1 (b) 2 (c) 3 (d) 4
5. $\lambda^{-1}(5) =$ (a) 0 (b) 1 (c) 2 (d) -1
6. Which of the following statements is not true? (a) I(n) is completely multiplicative (b) λ is multiplicative but not completely multiplicative (c) Only multiplicative function has dirichlet inverse (d) If f is multiplicative then f(1) = 1
7. Let f be any arbitrary function and g(x) > 0 for all $x \geq a$. Then $f(x)$ is asymptotic to g(x) if $\lim_{x\to\infty} \frac{f(x)}{g(x)} =$ (a) 0 (b) 1 (c) neither (a) nor (b) (d) none of the above
8. The number of lattice points in the region $\{x, y \in R : |x| \leq 2; |y| \leq 2\}$ (a) 4 (b) 8 (c) 16 (d) 25
9. $\lim_{x\to\infty} \sum_{n \leq x} \Lambda(n) =$ (a) 0 (b) 1 (c) logx (d) none of the above
10. $x - [x] =$ (a) $o\left(\frac{1}{x}\right)$ (b) $o\left(\frac{1}{x^2}\right)$ (c) $o(x)$ (d) none of the above

Part - B Answer (a) or (b) in each question $(5 \times 5 = 25\text{marks})$

11a) State and prove the division algorithm. State Euclidean algorithm (or)

b) (i) State and prove Euclid's lemma. (ii) State and prove the commutative, associative, and distributive properties of the gcd of (a, b)

12a) State and prove the relation between Euler totient function and the Mobius function. (or)

b) Prove that $\frac{n}{\varphi(n)} = \sum_{d|n} \frac{\mu^2(d)}{\varphi(d)}$.

13a) Define Mobius function. Is the Mobius function multiplicative or completely multiplicative? Justify your answer. (or)

b) Assume f is multiplicative. Prove that (i) $f^{-1}(n) = \mu(n) f(n)$ where n is square free. (ii) $f^{-1}(p^2) = (f(p))^2 - f(p^2)$ for every prime p. .

14a) State and prove the Euler summation formula (or)

b) If $\beta > 0$ and let $\delta = \max\{0, 1-\beta\}$. then if $x > 1$, prove that $\sum_{n \leq x} \sigma_{-\beta}(n) = \zeta(\beta+1)x + O(x^\delta)$ $if\ \beta \neq 1$; $= \zeta(2)x + O(\log x)$ $if\ \beta = 1$

15a) Prove that (i) for all $x \geq 1$; $\left|\sum_{n \leq x} \frac{\mu(n)}{n}\right| \leq 1$ with equality holding only if $x < 2$. (ii) For $x \geq 2$ prove that $\sum_{p \leq x} \left[\frac{x}{p}\right] \log p = x \log x + O(x)$ (08)

b) State and prove Abel's identity.

Part - C Answer (a) or (b) in each question $(5 \times 8 = 40\text{marks})$

16a) Given integers a and b, prove that there is a unique number d with the following properties: (a) $d \geq 0$; (b) $d|a$ and $d|b$; (c) $e|a$ and $e|b \Rightarrow e|d$ (05)

b) i)State and prove the fundamental theorem of arithmetic. ii) prove that $n^4 + 4$ is composite

17a) For $n \geq 1$ prove that $\varphi(n) = n \prod_{p|n}(1 - \frac{1}{p})$. If the same primes divide m and n, then $n\varphi(m) = m\varphi(n)$ (05)

b) Define the Euler totient function and prove that if $n \geq 1$ $\sum_{d|n} \varphi(d) = n$

18a) State and prove the associative property relating $\circ$ and $*$ and the generalized inversion formula and generalized Mobius inversion formula. (05)

b) i) let f be multiplicative. Then Prove that f is completely multiplicative if and only if $f^{-1}(n) = \mu(n) f(n)$ for all $n \geq 1$ ii) Find the inverse of the Euler totient function and Liuvelle's function.

19a) If $x \geq 1$ prove that (i) $\sum_{n \leq x} 1/n = \log x + C + O\left(\frac{1}{x}\right)$ (ii) $\sum_{n \leq x} 1/n^s = \frac{x^{1-s}}{1-s} + \zeta(s) + O(x^{-s})$ $if\ s > 0\ and\ s \neq 1$ (iii) $\sum_{n > x} \frac{1}{n^s} = O(x^{1-s})$ $if\ s > 1$ (iv) $\sum_{n \leq x} n^{\alpha} = \frac{x^{\alpha+1}}{\alpha+1} + O(x^{\alpha})$ $if\ \alpha \geq 0$ (08)

b) (i) Two lattice points (a, b) and (m, n) are mutually visible if and only if a-m and b-n are relatively prime (ii) The set of lattice points visible from the origin has density $\frac{6}{\pi^2}$

20a) Prove that (i) $\lim_{x \to \infty} \frac{\pi(x) \log x}{x} = 1 \Leftrightarrow$ (ii) $\lim_{x \to \infty} \frac{\vartheta(x)}{x} = 1 \Leftrightarrow$ (iii) $\lim_{x \to \infty} \frac{\psi(x)}{x} = 1$ (08)

b) For every integer $n \geq 2$ prove that $\frac{1}{6} \frac{n}{\log n} < \pi(n) < 6 \frac{n}{\log n}$